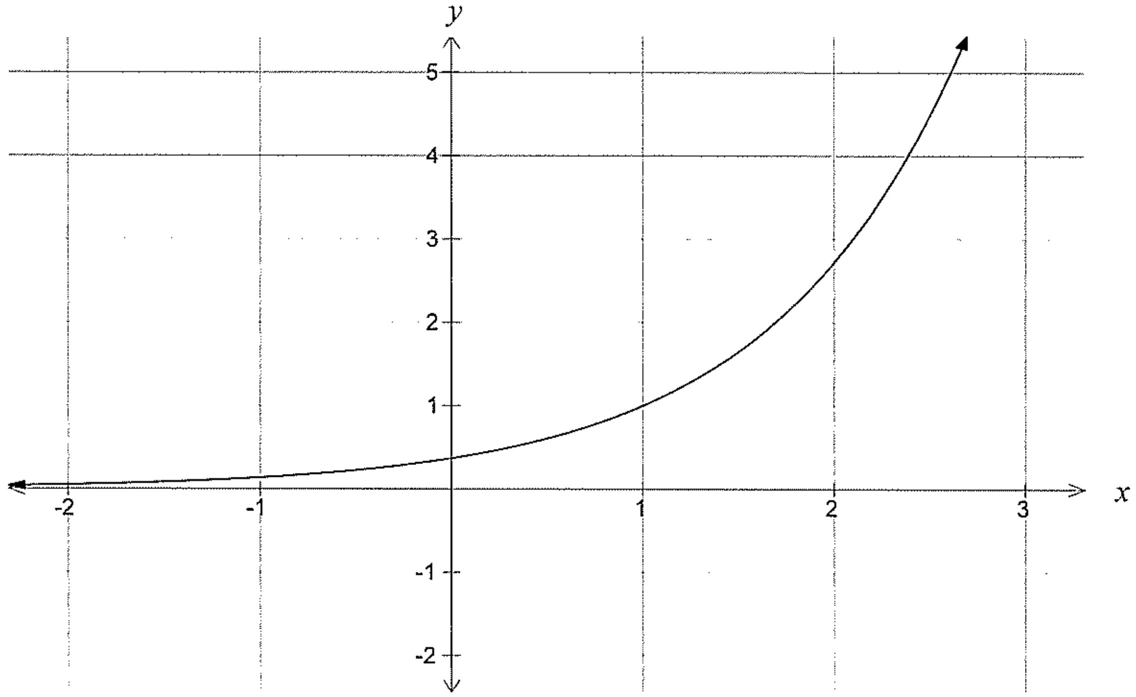
Question 4 **(4 marks)**

The graph of $y = e^{(x-1)}$ is shown below.



Calculate the exact area between the graphs of $y = e^{(x-1)}$, $y = 2 - x$ and the two axes.

Section One: Calculator-free (40 Marks)

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 (6 marks)

Differentiate the following with respect to x, without simplifying.

(a) $f(x)=\dfrac{4x+1}{\sqrt{x^2+1}}$ (2 marks)

(b) $g(x)=xe^{x^2+1}$ (2 marks)

(c) $h(x)=\int_x^1 (1+2t)^2\,dt$ (2 marks)

Question 4 **(4 marks)**

Let $f(x) = e^x$ and $g(x) = \sqrt{1-x}$.

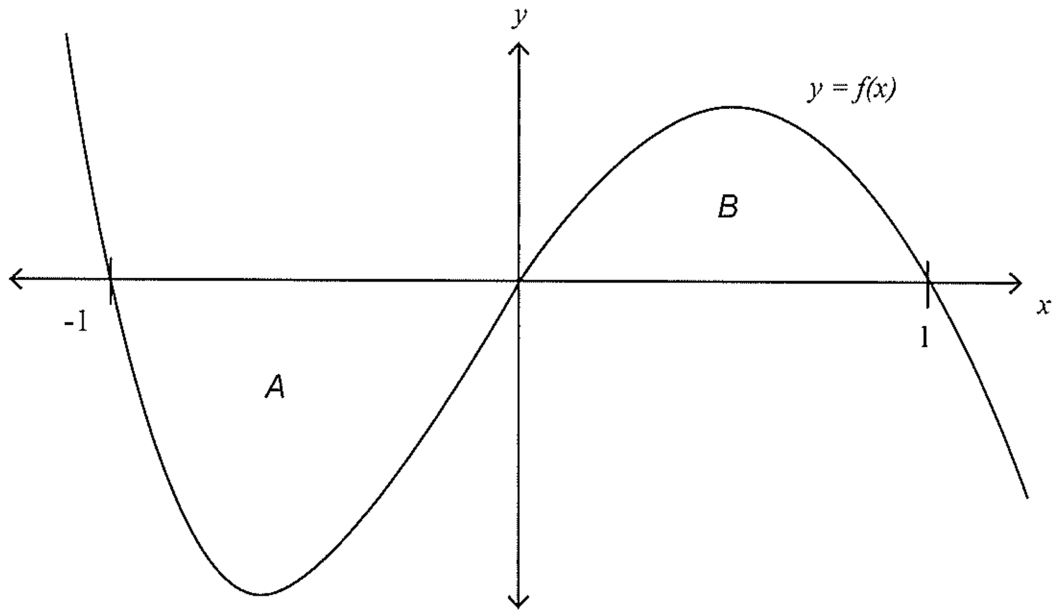
(a) Determine expressions for $f(g(x))$ and $g(f(x))$. (2 marks)

(b) Determine the range of $f(g(x))$. (1 mark)

(c) Determine the domain of $g(f(x))$. (1 mark)

Question 8 **(5 marks)**

Part of the graph of $y = f(x)$ is shown below. The areas of the bounded regions *A* and *B* are 7 and 4 square units respectively.



(a) Evaluate $\int_0^1 f(-x)dx$ (2 marks)

(b) Evaluate $\int_{-1}^1 (2 - f(x))dx$ (3 marks)

End of questions

Question 11 **(6 marks)**

A radioactive substance is decaying exponentially, according to the formula

$A(t) = A(0)e^{-kt}$, where $A(t)$ kg is the amount at time t years.

(a) Determine k, correct to 4 significant figures, given that the half-life of the substance is 12 years. (2 marks)

A second radioactive substance is also decaying exponentially, according to the formula

$B(t) = B(0)e^{-0.04t}$, where $B(t)$ kg is the amount at time t years.

(b) Which of these substances is decaying faster? Justify your answer briefly. (1 mark)

At a certain location there was exactly the same amount of these two substances at the beginning of the year 2010.

(c) In what year will the ratio of the amount of one of these substances to the other be 2:1? (3 marks)

Question 16 **(7 marks)**

In a particular valley, only two types of plants grow: Blues and Reds. Today there are 130 hectares of Blue plants and 40 hectares of Red plants.

Due to a disease, that affects only Blue plants, the number of hectares of Blue plants, B, can be described by the differential equation $\frac{dB}{dt} = -0.15B$, where t is measured in years.

(a) In how many years will the number of hectares of Blue plants halve? (3 marks)

(b) The number of hectares of Red plants, R, grows exponentially. Find the percentage growth rate in R if, after seven years, the total number of hectares of Blue and Red plants is the same as today. (4 marks)

DO NOT WRITE IN THIS AREA

Question 10 **(7 marks)**

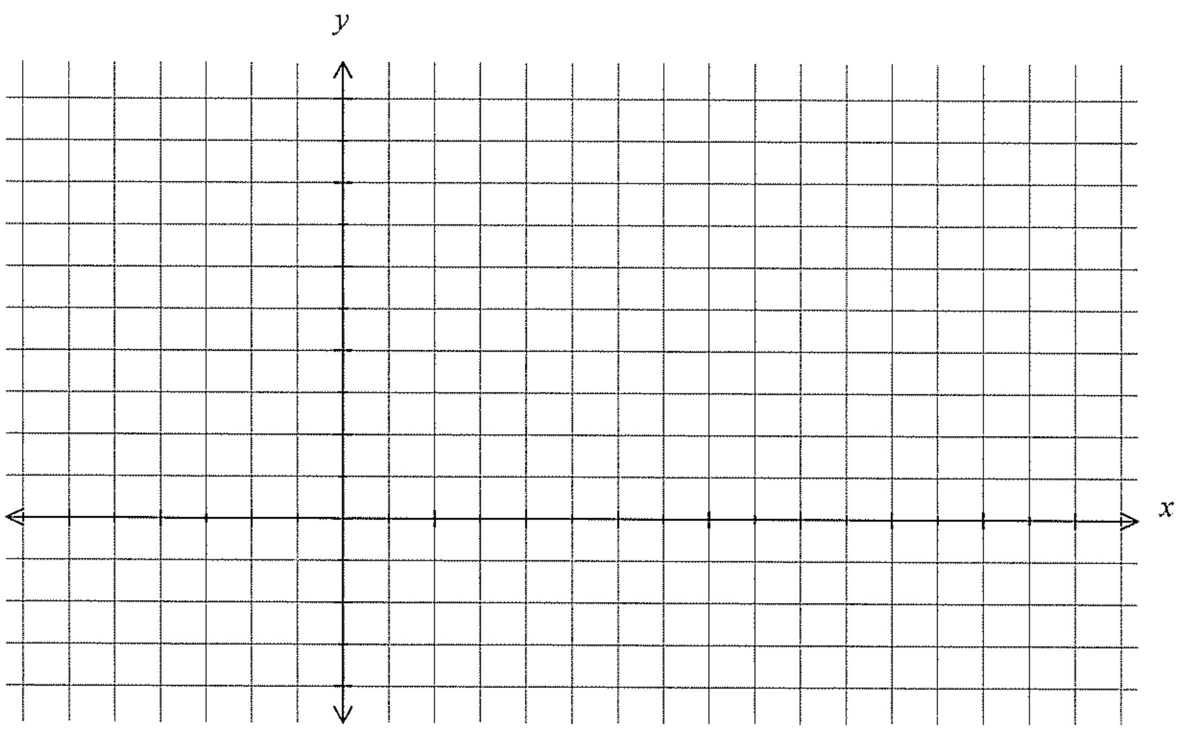
The function $y = f(x)$ is given by $f(x) = e^{ax-2}$, where a is a constant.

When $y = f(x)$ and $y = f^{-1}(x)$ are plotted on the same set of axes, they intersect at a point where $x = 3$.

(a) Write down the exact value of y at this point of intersection. (1 mark)

(b) Find the value of a correct to 2 decimal places. (2 marks)

(c) Draw $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing the coordinates of the point(s) of intersection. (4 marks)



Question 11 **(5 marks)**

When an amount $\$A$ is invested at an interest rate of $r\%$ per annum, compounded n times per year, the value $\$V$ of the investment after one year is given by $V = A\left(1+\frac{r}{100n}\right)^n$.

Kelvin invests $6000 at 8% per annum interest for one year.

(a) What is the value of the investment at the end of the year if interest is compounded twice per year? (1 mark)

(b) If interest is compounded monthly, by what percentage does the investment increase over the course of the year? (2 marks)

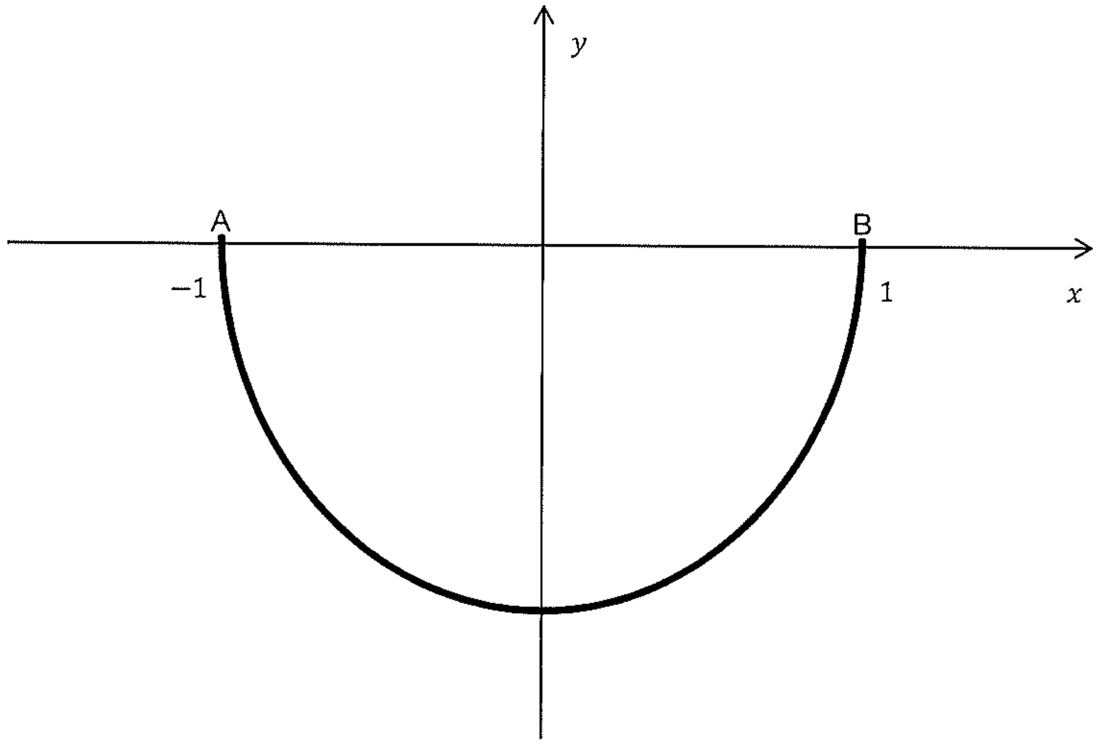
(c) If the investment were to be compounded more frequently, could the value of Kelvin's investment rise above $6500 at the end of the year? Justify your answer. (2 marks)

Question 18 **(7 marks)**

A cable hanging between two points $A(-1,0)$ and $B(1,0)$ lies on the curve

$$y = e^{cx} - d + e^{-cx},$$

where c and d are positive constants.



(a) Show that $d = e^{c} + e^{-c}$. (1 mark)

(b) Use calculus to show that the lowest point on the cable occurs where it crosses the y-axis, that is, where $x = 0$. (3 marks)

(c) The length s of the curve $y = f(x)$, between the limits $x = a$ and $x = b$, is given by the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Use this formula to determine the length of the cable if the lowest point of the cable is 10 units below the level of the supports A and B. (3 marks)

Section Two: Calculator-assumed (80 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 8 **(5 marks)**

Radium decays at a rate proportional to its present mass; that is, if $Q(t)$ is the mass of radium present at time t, then $\frac{dQ}{dt} = kQ$.

It takes 1600 years for any mass of radium to reduce by half.

(a) Find the value of k. (3 marks)

(b) A factory site is contaminated with radium. The mass of radium on the site is currently five times the safe level. How many years will it be before the mass of radium reaches the safe level? (2 marks)

Question 18 **(8 marks)**

A model for a population, P, of numbats is

$P = \dfrac{900}{3+2e^{-t/4}}$, where t is the time in years from today.

(a) What is the population today? (1 mark)

(b) What does the model predict that the eventual population will be? (1 mark)

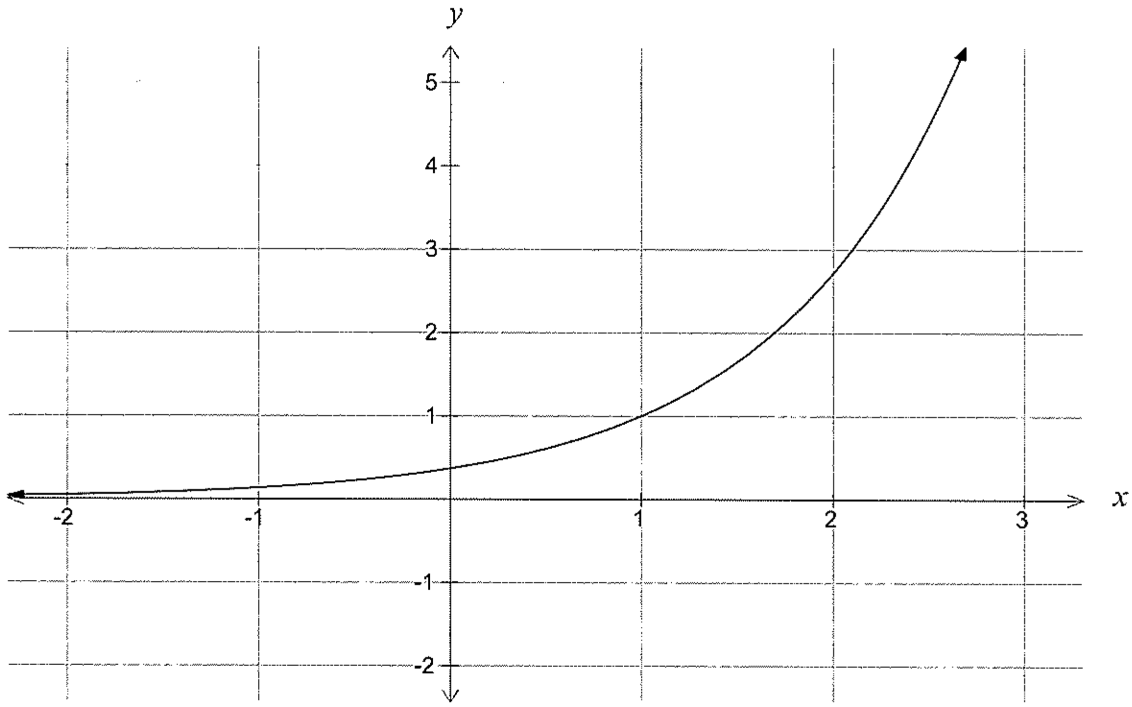
Question 18 (continued)

(c) By first expressing $e^{-t/4}$ in terms of P, or otherwise, show that P satisfies the differential equation $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$. (4 marks)

(d) What is the instantaneous percentage annual rate of growth today? (2 marks)

Question 4 **(4 marks)**

The graph of $y = e^{(x-1)}$ is shown below.



Calculate the exact area between the graphs of $y = e^{(x-1)}$, $y = 2 - x$ and the two axes.

Solution
$A = \int_0^1 e^{(x-1)}dx + \int_1^2 (2-x)dx$ i.e. $A = \left[e^{(x-1)}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$ i.e. $A = 1 - e^{-1} + (4-2) - \left(2 - \frac{1}{2}\right)$ i.e. $A = \frac{3}{2} - \frac{1}{e}$
Specific behaviours
✓ recognises that the area is the sum of two integrals   ✓ identifies the correct limits for each integral   ✓ correctly integrates   ✓ substitutes and simplifies

Section One: Calculator Free **(40 marks)**

Question 1 **(6 marks)**

Differentiate the following with respect to x, without simplifying.

(a) $f(x)=\dfrac{4x+1}{\sqrt{x^2+1}}$ (2 marks)

Solution
$f'(x)=\dfrac{\sqrt{x^2+1}\times 4-(4x+1)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{x^2+1}$ or $f'(x)=\dfrac{4}{\sqrt{x^2+1}}+(4x+1)(-\frac{1}{2})(x^2+1)^{-\frac{3}{2}}(2x)$
Specific Behaviours
✓ differentiates denominator of $f(x)$ correctly ✓ combines component expressions using the quotient or product rule correctly

(b) $g(x)=xe^{x^2+1}$ (2 marks)

Solution
$g'(x)=e^{x^2+1}+xe^{x^2+1}(2x)$
Specific Behaviours
✓ differentiates correctly e^{x^2+1} ✓ combines component expressions using the product rule correctly

(c) $h(x)=\int_x^1(1+2t)^2\,dt$ (2 marks)

Solution
$h'(x)=-(1+2x)^2$ or $h(x)=\int_x^1(1+2t)^2\,dt=\left[\frac{1}{6}(1+2t)^3\right]_x^1=\frac{1}{6}(3)^3-\frac{1}{6}(1+2x)^3$ so $h'(x)=-\frac{1}{6}(3)(2)(1+2x)^2=-(1+2x)^2$
Specific Behaviours
✓ applies Fundamental Theorem of Calculus to obtain $(1+2x)^2$ ✓ recognises that the limits of integration imply that $-(1+2x)^2$ is the solution or ✓ integrates correctly ✓ differentiates correctly

Question 4 **(4 marks)**

Let $f(x) = e^x$ and $g(x) = \sqrt{1-x}$.

(a) Determine expressions for $f(g(x))$ and $g(f(x))$. (2 marks)

Solution
$f(g(x)) = e^{\sqrt{1-x}}$ $g(f(x)) = \sqrt{1-e^x}$
Specific behaviours
✓ determines correct expression for $f(g(x))$ ✓ determines correct expression for $g(f(x))$

(b) Determine the range of $f(g(x))$. (1 mark)

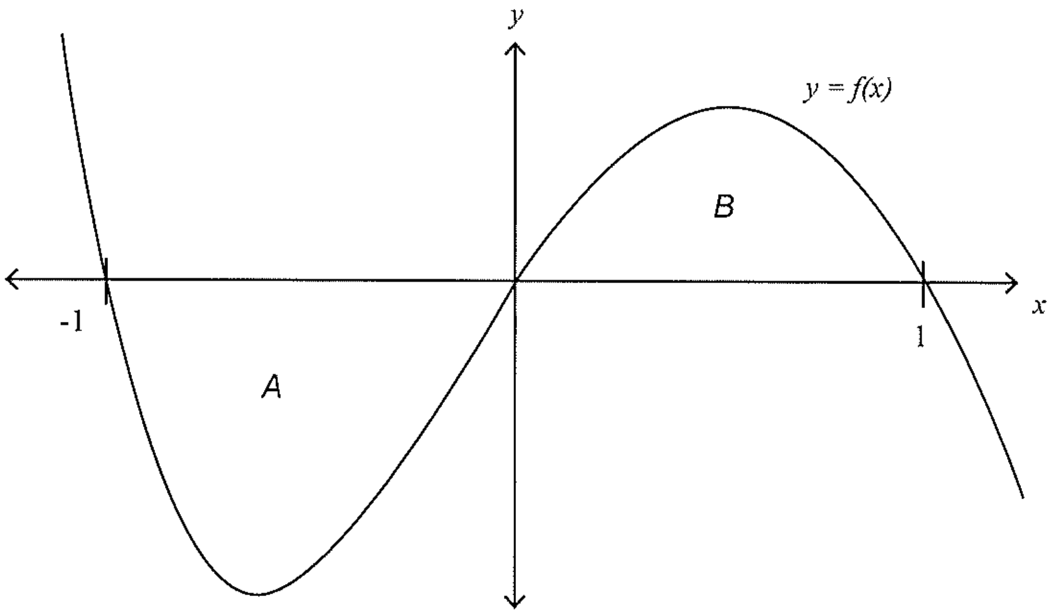
Solution
Since $\sqrt{1-x}$ can be any positive number the range of $f(g(x))$ is the set of all numbers e^y where $y \geq 0$. Since $e^0 = 1$, the range is the interval $[1,\infty)$, or $1 \leq y < \infty$.
Specific behaviours
✓ determines the appropriate interval

(c) Determine the domain of $g(f(x))$. (1 mark)

Solution
$\sqrt{1-e^x}$ is defined provided that $1-e^x \geq 0$, i.e. $1 \geq e^x$, i.e. $x \leq 0$ So the domain of $g(f(x))$ is the interval $x \leq 0$, or $(-\infty, 0]$.
Specific behaviours
✓ correctly determines the appropriate interval

Question 8 **(5 marks)**

Part of the graph of $y = f(x)$ is shown below. The areas of the bounded regions *A* and *B* are 7 and 4 square units respectively.



(a) Evaluate $\int_0^1 f(-x)dx$ (2 marks)

Solution
The graph of $y = f(-x)$ is the reflection of the above graph in the y-axis. The required integral is the signed area $-A$, i.e. -7.
Specific behaviours
✓ determines correctly the magnitude of the integral ✓ determines correctly the sign of the integral

(b) Evaluate $\int_{-1}^1 (2 - f(x))dx$ (3 marks)

Solution
$\int_{-1}^{1}(2-f(x))dx = \int_{-1}^{1}2\,dx - \int_{-1}^{1}f(x)dx = 4-(-7+4) = 7$
Specific behaviours
✓ identifies correctly the two components of the integral ✓✓ determines the two integrals

Question 11 **(6 marks)**

A radioactive substance is decaying exponentially, according to the formula

$$A(t) = A(0)e^{-kt}, \text{ where } A(t) \text{ kg is the amount at time } t \text{ years.}$$

(a) Determine k, correct to 4 significant figures, given that the half-life of the substance is 12 years. (2)

Solution	
Since $A(12) = \frac{1}{2}A(0)$, $\frac{1}{2} = e^{-12k}$ So $k = 0.05776$ (from calculator)	
Specific Behaviours	
Obtains answer	✓
Gives 4 significant figures	✓

A second radioactive substance is also decaying exponentially, according to the formula

$$B(t) = B(0)e^{-0.04t}, \text{ where } B(t) \text{ kg is the amount at time } t \text{ years.}$$

(b) Which of these substances is decaying faster? Justify your answer briefly. (1)

Solution	
A is decaying faster because $k > 0.04$	
Specific Behaviours	
Obtains answer with correct reasoning	✓

At a certain location there was exactly the same amount of these two substances at the beginning of the year 2010.

(c) In what year will the ratio of the amount of one of these substances to the other be 2:1?(3)

Solution	
Since $A(t) = B(t)/2$, $A(0)e^{-0.05776t} = B(0)e^{-0.04t}/2$ (*) But $A(0) = B(0)$, so $e^{-0.05776t} = e^{-0.04t}/2$, i.e. $\frac{e^{-0.04t}}{e^{-0.05776t}} = 2$ i.e. $\frac{e^{-0.04t}}{e^{-0.05776t}} = e^{0.05776t-0.04t} = 2$ So $t = 39.03$ (from calculator) So the year will be 2049	
Specific Behaviours	
Obtains equation (*)	✓
Obtains value of t	✓
Interprets answer to obtain year	✓

Question 16 **(7 marks)**

In a particular valley, only two types of plants grow: Blues and Reds. Today there are 130 hectares of Blue plants and 40 hectares of Red plants.

Due to a disease, that affects only Blue plants, the number of hectares of Blue plants, B, can be described by the differential equation $\frac{dB}{dt} = -0.15B$, where t is measured in years.

(a) In how many years will the number of hectares of Blue plants halve? (3 marks)

Solution
$B = 130e^{-0.15t}$
Thus for the half-life, $\frac{1}{2} = e^{-0.15t}$
$t = 4.621$
The number of hectares of Blue plants will halve in approximately 4.6 years
Specific behaviours
✓ correctly expresses B in terms of t
✓ states the correct half-life equation
✓ calculates the final answer

(b) The number of hectares of Red plants, R, grows exponentially. Find the percentage growth rate in R if, after seven years, the total number of hectares of Blue and Red plants is the same as today. (4 marks)

Solution
$R = 40e^{kt}$
When $t = 7$:
$130e^{-0.15\times 7} + 40e^{7k} = 130 + 40$
Solve using CAS to find $k = 0.162$
The percentage growth rate of R is 16.2%
Specific behaviours
✓ correctly expresses R in terms of t
✓ states the correct equation
✓ calculates the value for k
✓ states the percentage growth rate for R

Question 10 **(7 marks)**

The function $y = f(x)$ is given by $f(x) = e^{ax-2}$, where a is a constant.

When $y = f(x)$ and $y = f^{-1}(x)$ are plotted on the same set of axes, they intersect at a point where $x = 3$.

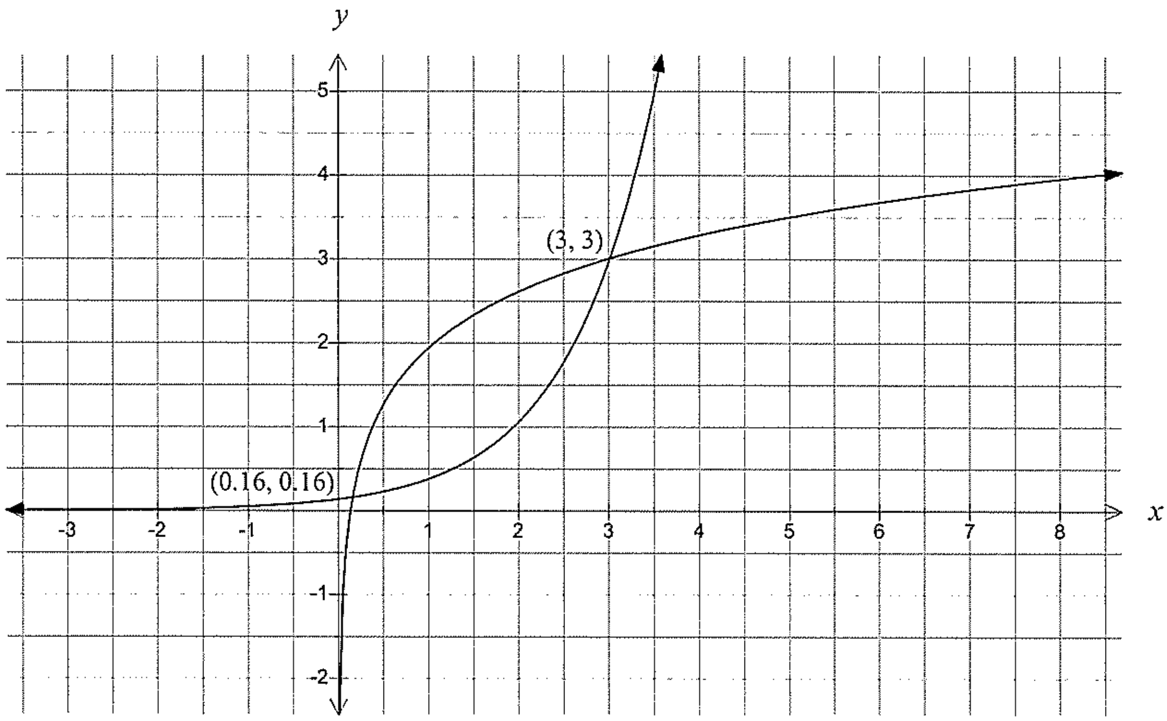
(a) Write down the exact value of y at this point of intersection. (1 mark)

Solution
$y = 3$
Specific behaviours
✓ recognises that a function and its inverse intersect on the line $y = x$

(b) Find the value of a correct to 2 decimal places. (2 marks)

Solution
Point $(3,\ 3)$ lies on the graph of $y = f(x)$ i.e. $e^{3a-2} = 3$ Solve using a CAS i.e. $a = 1.03$
Specific behaviours
✓ sets up the equation using the point $(3,\ 3)$ ✓ solves for a

(c) Draw $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing the coordinates of the point(s) of intersection. (4 marks)

Solution


Specific behaviours
✓ sketches the graph of $y = f(x)$ **And either:** ✓ uses a CAS to determine $y = f^{-1}(x) = \dfrac{\ln x + 2}{1.03}$ ✓ sketches the graph of $y = f^{-1}(x)$ ✓ identifies the point of intersection $(0.16,\ 0.16)$. Note: $(3,3)$ previously identified **Or:** ✓✓sketches the graph of $y = f^{-1}(x)$ as a reflection of $y = f(x)$, with correct asymptote and correct shape ✓ identifies the point of intersection $(0.16,\ 0.16)$. Note: $(3,3)$ previously identified

Question 11 **(5 marks)**

When an amount $\$A$ is invested at an interest rate of $r\%$ per annum, compounded n times per year, the value $\$V$ of the investment after one year is given by $V = A\left(1+\frac{r}{100n}\right)^n$.

Kelvin invests $6000 at 8% per annum interest for one year.

(a) What is the value of the investment at the end of the year if interest is compounded twice per year? (1 mark)

Solution
$V = 6000\left(1+\frac{8}{2\times 100}\right)^2 = 6489.60$ So the value at the end of the year is $6489.60
Specific behaviours
✓ evaluates correct value of investment

(b) If interest is compounded monthly, by what percentage does the investment increase over the course of the year? (2 marks)

Solution
$V = 6000\left(1+\frac{8}{12\times 100}\right)^{12} = 6498.00$ So the percentage growth is $\frac{6498-6000}{6000}\times 100 = 8.3\%$
Specific behaviours
✓ calculates correct value of investment after one year ✓ calculates correct percentage increase

(c) If the investment were to be compounded more frequently, could the value of Kelvin's investment rise above $6500 at the end of the year? Justify your answer. (2 marks)

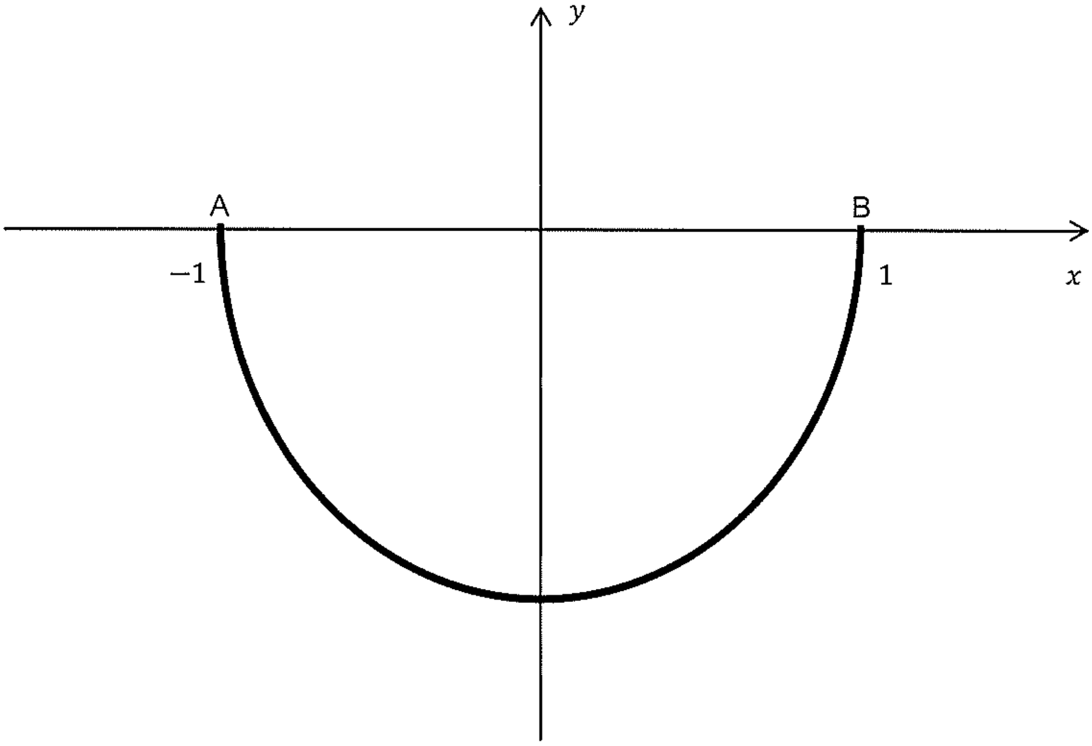
Solution
The maximum occurs if interest is compounded 'continuously'. In which case the value after one year is given by $V = 6000e^{0.08} = 6499.72$ Therefore the value can't rise above $6500 by more frequent compounding.
Specific behaviours
✓ calculates the correct limiting value ✓ interprets the limiting value

Question 18 **(7 marks)**

A cable hanging between two points $A(-1,0)$ and $B(1,0)$ lies on the curve

$$y = e^{cx} - d + e^{-cx},$$

where c and d are positive constants.



(a) Show that $d = e^c + e^{-c}$. (1 mark)

Solution
$y = 0$ when $x = 1$, and so $0 = e^c - d + e^{-c}$ Hence $d = e^c + e^{-c}$.
Specific behaviours
✓ substitutes (–1,0) or (1,0) into equation

(b) Use calculus to show that the lowest point on the cable occurs where it crosses the y-axis, that is, where $x = 0$. (3 marks)

Solution

The minimum will occur when $\frac{dy}{dx} = 0$

Now $\frac{dy}{dx} = ce^{cx} - ce^{-cx} = c(e^{cx} - e^{-cx}) = 0$

This occurs when $e^{cx} = e^{-cx}$.

Dividing both sides by e^{-cx}, we have $e^{2cx} = 1$

So $x = 0$.

Specific behaviours

✓ calculates derivative

✓ simplifies to find that $e^{cx} = e^{-cx}$

✓ solves for x

(c) The length s of the curve $y = f(x)$, between the limits $x = a$ and $x = b$ is given by the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Use this formula to determine the length of the cable if the lowest point of the cable is 10 units below the level of the supports A and B. (3 marks)

Solution

Since the minimum occurs at $x = 0$, $-10 = e^0 - e^c - e^{-c} + e^0$, i.e. $e^c + e^{-c} = 12$.
From the calculator: $c = 2.47789$
So $\frac{dy}{dx} = 2.47789(e^{2.477889x} - e^{-2.477889x})$,

So $s = \int_{-1}^{1} \sqrt{1 + \left(2.47789(e^{2.47789x} - e^{-2.47789x})\right)^2}\, dx = 20.2712$

Specific behaviours

✓ states equation which can be used to find c

✓ solves for c

✓ calculates length of curve

Section Two: Calculator-assumed **(80 Marks)**

Question 8 **(5 marks)**

Radium decays at a rate proportional to its present mass; that is, if $Q(t)$ is the mass of radium present at time t, then $\frac{dQ}{dt} = kQ$.

It takes 1600 years for any mass of radium to reduce by half.

(a) Find the value of k. (3 marks)

Solution
$Q(t) = Ae^{kt}$ $\frac{1}{2} = e^{1600k}$ Hence k = -0.000433 (Accept $\frac{-2\log^2}{1600}$)
Writes the specific behaviours
✓ writes the exponential decay equation ✓ writes an equation for the half-life of radium ✓ solves for k

(b) A factory site is contaminated with radium. The mass of radium on the site is currently five times the safe level. How many years will it be before the mass of radium reaches the safe level? (2 marks)

Solution
Let S be the safe level of radium. Then the initial value satisfies $A = 5S$ i.e. $\frac{1}{5} = e^{\frac{\ln 0.5}{1600}t}$ t = 3715 (Accept 3715 or 3716) It will be 3716 years before the site is safe Or $\frac{1}{5} = e^{-0.000433t}$ t = 3716.95 (Accept 3716 or 3717) It will be 3717 years before the site is safe
Specific behaviours
✓ correctly expresses A in terms of S (or correct ratio) ✓ solves for t

Question 18 **(8 marks)**

A model for a population, P, of numbats is

$P = \dfrac{900}{3+2e^{-t/4}}$, where t is the time in years from today.

(a) What is the population today? (1 mark)

Solution
Population today = $\dfrac{900}{3+2} = 180$
Specific behaviours
✓ sets $t = 0$ and solves for P

(b) What does the model predict that the eventual population will be? (1 mark)

Solution
Eventual population = $\dfrac{900}{3} = 300$
Specific behaviours
✓ lets $t \to \infty$ and solves for P

(c) By first expressing $e^{-t/4}$ in terms of P, or otherwise, show that P satisfies the differential equation $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$. (4 marks)

Solution

$$e^{-t/4} = \frac{450}{P} - \frac{3}{2}$$

Hence $-\frac{e^{-t/4}}{4} = -\frac{450}{P^2} \times \frac{dP}{dt}$

i.e. $\frac{1}{4} \times \left(\frac{450}{P} - \frac{3}{2}\right) = \frac{450}{P^2} \times \frac{dP}{dt}$

Hence $\frac{dP}{dt} = \frac{P^2}{4 \times 450} \times \left(\frac{450}{P} - \frac{3}{2}\right)$

i.e. $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$

Specific behaviours

- ✓ correctly rearranges the equation
- ✓ differentiates $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ implicitly with respect to t
- ✓ substitutes $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ to give an equation for $\frac{dP}{dt}$ involving P only
- ✓ rearranges and simplifies

(d) What is the instantaneous percentage annual rate of growth today? (2 marks)

Solution

The instantaneous rate of change today = $\frac{dP}{dt}$ when t = 0 and P = 180

i.e. $\left.\frac{dP}{dt}\right|_{t=0} = 18$

Hence the instantaneous percentage rate of growth today = $\frac{18}{180} \times 100 = 10\%$

Specific behaviours

- ✓ calculates the instantaneous rate of change today
- ✓ calculates the required percentage